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THE HISTORY OF DETERMINANTS.

The Theory of Determinants in the Historical Order of its Development. Part I. Determinants in General. By Thomas Muir, M.A., LL.D., F.R.S.E. (London: Macmillan and Co., 1890.)

THE theory of determinants is in that borderland which separates the "pass" from the "honour" student of pure mathematics. In elementary text-books the subject is rarely more than introduced for the purpose of representing some result of geometry or analysis in a convenient, beautiful, and suggestive form. The essential properties of a determinant are not set forth, but the student is perhaps referred for further information to one or other of the two excellent treatises which are already at our disposal in the English language, viz. those of Mr. Muir and of Mr. R. F. Scott. The value of the idea thus given to a student of the shape and convenient use of a great analytical implement is beyond all question. His imagination and curiosity are alike excited, and the "trick" possibly prevents his passage through life under the delusion that all mathematics are comprised within the covers of the school-books. The honour student, as a matter of course, reads some work on the subject, and is as surely enchanted. He cannot fail to recognize the power and beauty of the notation. He observes that the object of his study is constructive in its nature. He becomes convinced that pure mathematics is one of the fine arts, and just as a beautiful picture gives pleasure to one who understands painting, just as a fine piece of sculpture delights one who understands modelling, so he sees unfolded to his intellectual eye an exquisite example of constructive art, which his previous mathematical reading has fitted him to understand and appreciate, and to regard as a beautiful object of contemplation. The theory of determinants is one of the most artistic subdivisions of mathematical science, and accordingly has never wanted enthusiastic admirers. It is most gratifying to find such an authority as Mr. Muir devoting his leisure to its historical development. Any mathematician taking up this volume would anticipate a treat, and he would not be disappointed. In this first instalment the reader is taken from Leibnitz (1693) to Cauchy (1841). Mr. Muir assigns a chief place of honour to Vandermonde (1771), who, in his "Mémoire sur l'Élimination" (Hist. de l'Acad. Roy. des Sciences), denoted a function formed from the coefficients of a set of linear equations by a symbolism which is at once recognized as a condensed form of the determinant matrix of the present day. He was the first to give a connected exposition of the theory, and to give the true fundamental properties of the new functions. His notation, moreover, was exceedingly good, and much superior to that adopted by some subsequent writers who overlooked or neglected his important work.

Vandermonde has also recently received justice, long denied him, in other branches of analysis, and there is now no doubt that the value and originality of his work entitle him to a place in the ranks of the mathematical pioneers of his time. Up to the close of the eighteenth century the most noteworthy additions were made by Laplace, Lagrange, and Bezont. We find that Lagrange knew that the discriminant of a binary quantic of the second order is an invariant of the linear transformation. He did not, however, express either the discriminant or the determinant of transformation in a determinant form. The author critically examines the claim of Laplace to be the discoverer of the expansion theorem. He finds that although the case in which as many as possible of the factors of the terms of the expansion are of the second degree had already been given by Vandermonde, and Laplace himself did not give a statement of the rule suited for general application, the claim can in the main be upheld. Hindenburg (1784) and Rothe (1800) took up the subject in Germany, and between them constructed an elementary theory of permutations. Rothe, it is interesting to observe, employed a graphical method which will remind the reader of Prof. Sylvester's modern constructive theory of partitions. Gauss (1801) followed, and then we come to the important papers of Binet (1811) and Cauchy (1812). These memoirs establish the multiplication theorem in its full generalization. The method adopted by Binet may be described as that of symmetric functions, which he uses freely. He employs identities of the type

$$\Sigma ab'c'' = \Sigma a\Sigma b\Sigma c + 2\Sigma abc - \Sigma a\Sigma bc - \Sigma b\Sigma ca - \Sigma e\Sigma ab,$$

having reference to several systems of quantities. He was not, however, sufficiently acquainted with the theory of such functions; and was unable to supply rigid proofs of the theorems in determinants which, from his point of view, rested upon these identities. Nowadays, the identity in question will be recognized as the expression of an "elementary" symmetric function (single unitary, and having three parts in its partition) by means of symmetric functions each of which is expressible symbolically by a partition composed of one tripartite part. The law of the coefficients, undivulged by Binet, is now perfectly well known. It is, in fact, an easy generalization of the law by which, in the case of a single system of quantities, the elementary symmetric functions are expressed as functions of the sums of powers of the quantities. Cauchy at the same date (1812) introduced the idea of "fonctions symétriques alternées," which led him to a new symbolic definition of a determinant and to many valuable results. Mr. Muir devotes several pages to an examination of Cauchy's title to share with Binet the credit of the generalized multiplication theorem. He gives his decision against Cauchy, and probably the reader who closely follows the argument will find himself in accord with the historian. Notwithstanding this conclusion, Cauchy's memoir is excellent in quality and abundant in quantity; he "opened up a whole avenue of fresh investigation, and one cannot but assign to him the place of highest honour among all the workers from 1693 to 1812."

A retrospect is given of the period 1693–1812 accompanied by an interesting tabular record. As a means of reference the work appears to be absolutely perfect. Each new result as it appears is marked in Roman figures; and if the same result be obtained differently, or be generalized by a subsequent investigator, the same Roman figure is employed, followed by an Arabic numeral. It is found that to this point nearly every important advance is due to French mathematicians.

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During the second period (1813–1841) the chief names are Cauchy, Schweins, Jacobi, and Sylvester; to these may be added Desnanot and Scherk, to whom fresh departures, of less extent, are due. Schweins himself may be said to have been discovered by Mr. Muir. This author (1825) deals with the subject under the title "Theorie der Producte mit Versetzungen." He made a notable generalization described as the transformation of an aggregate of products of pairs of determinants into another aggregate of similar kind. He further discussed special forms, and, it is clear, possessed a firm grasp of his subject.

The work of Jacobi and Sylvester, and the further work of Cauchy will be more or less familiar to mathematicians. Germany has passed to the first place; and the occurrence of Sylvester's name in the history marks a revival of learning in pure mathematics in England.

The volume is remarkable for the study it presents in nomenclature and notation. There is an extraordinary variety in the symbolism. It is easy to observe the distinctive characters of French and German notation that are so marked at the present day. It is well known that much lies in an appropriate notation. Every young mathematician with a predilection for original work should read this book, to note the power of suitable symbols, to grasp the reason of their power, and, above all, to see what to avoid.

The book also points a moral which is not far to seek. It would be easy to pick out many such phrases as, "was acquainted with the writings of very few of his predecessors"; "was unaware apparently of the existence of a theory of determinants"; "hasty, if not contemptuous, disregard of historical research." To this tendency to work on without proper attention to previous work is doubtless due in some measure the unfortunate multiplication of names and symbolisms which is so perplexing and irritating to a reader. This failure to collaborate with others can only retard the progress of the science. It is perfectly true that great original thinkers, like Gauss, Cauchy, Jacobi, and Sylvester, may take liberties of this kind; and the fact of their doing so may even be beneficial to the subject, as resulting in memoirs of more unfettered originality. But this is not so in the case of lesser men. In taking leave of this fascinating history one can look forward to Part II. with sincere pleasure, which is not diminished by the knowledge that the later developments have been largely due to our countrymen. We have yet to see Sylvester's most powerful investigations, and all Cayley's researches; and, finally, the successive steps by which the lofty heights of the theory of matrices and the theory of multiple algebra (involving the generalization of quaternions) have been attained.

P. A. M.

THE EVOLUTION OF MAN.

Anthropogenie, oder Entwickelungsgeschichte des Menschen. By Prof. Ernst Haeckel. Fourth Edition, revised and enlarged. Pp. i.–xxvi. and 1–906. (Leipzig: W. Engelmann, 1891.)

THE importance of the subject-matter of the book, the length of time that has elapsed since the appearance of the former editions, and the prominent position held by the author, seem to call for more detailed notice of this work than is usually accorded to a new edition.

The "Anthropogenie," which was first published in 1874, is the third of the series of books in which Prof. Haeckel has attempted to determine the laws governing the form, structure, and mutual relations of living things, and to establish the general principles of biological science. The first of these, under the title "Generelle Morphologie," appeared in 1866, almost simultaneously with the completion of Herbert Spencer's "Principles of Biology." It is a comprehensive and ambitious work, which, in its author's words,

> "constituted the first attempt to apply the general doctrine of development to the whole range of organic morphology, . . . and to introduce the Darwinian theory of descent into the systematic classification of animals and plants, and to found a 'natural system' on the basis of genealogy; that is, to construct hypothetical pedigrees for the various species of organisms."

It contains also the first systematic attempt to deal in detail with the ancestry of man, as regards the groups of animals lower than mammals. This is, perhaps, the most solid piece of work Prof. Haeckel has done; it contains much matter of great value, and discussions and speculations of extreme ingenuity and interest. Later discoveries have rendered much of it obsolete, but it still remains the most important work of its kind; and but for its somewhat cumbersome terminology, would be widely read even now. The "Natural History of Creation," first published in 1868, goes over a good deal of the same ground as the earlier work, but is written in a much more popular style, and aims at presenting in a form suited to the general reader the main arguments on which the Darwinian theory is based, together with a detailed application of the theory to the principal groups of animals, and an attempt to determine their mutual relations and lines of descent. The "Anthropogenie," the book now before us, is a more elaborate application of the same principles to the special problem of the evolution of man.

In the new edition the general arrangement remains much the same as before; but, in order to include the results of more recent investigations, a great part of the book has been re-written, and new chapters have been added on subjects that have attracted especial attention of late years, such as the Gastræa theory, the Cœlom theory, and the nature and origin of segmentation. A large number of new figures have been inserted, and the genealogical tables, for which Prof. Haeckel has a special fondness, have been greatly increased in number, and in elaboration of detail.

The book, which is adapted rather for the general reader than the scientific student, is written in an attractive and popular style, and presents the main facts of vertebrate embryology in an intelligible and well illustrated form. As might be expected from his former writings, the main feature of Prof. Haeckel's work is a detailed exposition and vigorous defence of what he has named "the fundamental law of biogenesis," better known in this country as the recapitulation theory, according to which the actual or ontogenetic development of an animal is a repetition of the ancestral or phylogenetic

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